

## Optimal Control for a Class of Distributed Parameter Systems Where the Cost Functions are Norms

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### ABSTRACT

Some problems in optimal control, for a class of linear distributed parameter systems, are posed. By using a theorem of approximation theory, a maximum principle, which is applicable to these optimal control problems, is given. It is shown, that for the problems posed in this paper, this maximum principle is an extension of Butkovskii's maximum principle and of Demyanov and Rubinov's theorem.

In addition, two optimal control problems, in linear lumped parameter systems are posed. For these problems it is shown that the maximum principle given in this paper, is an extension of Pontryagin's maximum principle.

### 1. INTRODUCTION

This paper deals with optimal control problems, in a class of linear distributed parameter systems, where the cost functions are norms. If the cost function is a differentiable functional on a suitable Banach space, then, Butkovskii's maximum principle ([1], [2]), can be applied for deriving necessary conditions upon the optimal controls. Furthermore, for the minimization of a differentiable convex functional, on a convex set, in a Banach space, one can use a theorem of Demyanov and Rubinov [3] for deriving necessary and sufficient conditions upon the optimal controls. For example see Yavin and Sivan [4]. Here, by using a theorem of Deutsch and Maserick [5], a maximum principle is given, which can be applied to optimal control problems in linear systems, where the cost function is a norm and the control functions belong to a convex set. Since this maximum principle is applicable to cases where the norms are not differentiable functionals, then it can be considered, at least for the problems posed in this paper, as an extension of Butkovskii's maximum principle ([1], [2]) and of Demyanov and Rubinov's theorem [3].

Also, for linear lumped parameter systems, where the cost functions are norms, it is shown that the maximum principle given in this paper is an extension of Pontryagin's maximum principle, see Rozonoer [6].

## 2. THE STATEMENT OF THE PROBLEMS

Let a one-dimensional linear distributed parameter system be given by:

$$y(x, t; u) = y_0(x, t) + \int_0^t h(x, t, s) u(s) ds, \quad 0 < x < 1, \quad 0 \leq t \leq T \quad (1)$$

where  $y_0 \in C[D_2]$  and  $h \in C[D_3]$  are given functions, and

$$D_2 \triangleq \{0 \leq x \leq 1, 0 \leq t \leq T\}, \quad \text{and} \quad D_3 \triangleq \{0 \leq x \leq 1, 0 \leq s, t \leq T\}.$$

$y(x, t; u)$  is the system's response, and  $u$  is the control function.

Given a set

$$\Omega_{p_1} \triangleq \left\{ u \mid \left[ \int_0^T |u(t)|^{p_1} dt \right]^{1/p_1} \leq 1 \right\},$$

where  $p_1$  is given,  $1 < p_1 \leq \infty$ ; the following problems are posed:

**Problem A.** Given a function  $y_d(x, t)$ ,  $y_d \in C[D_2]$ , the problem is to choose a control  $u^* \in \Omega_{p_1}$  such that:  $A(u) \geq A(u^*)$  for any  $u \in \Omega_{p_1}$ , where

$$A(u) \triangleq \left[ \int_0^T \int_0^1 |y_d(x, t) - y(x, t; u)|^{p_2} dx dt \right]^{1/p_2}, \quad (2)$$

and  $p_2$  and  $T$  are given numbers,  $T > 0$ ,  $1 \leq p_2 < \infty$ .

**Problem B.** Given a function  $y_d(x)$ ,  $y_d \in C[0, 1]$ , the problem is to choose a control  $u^* \in \Omega_{p_1}$  such that:  $B(u) \geq B(u^*)$  for any  $u \in \Omega_{p_1}$ , where

$$B(u) \triangleq \left[ \int_0^1 |y_d(x) - y(x, T; u)|^{p_2} dx \right]^{1/p_2}, \quad (3)$$

and  $p_2$  and  $T$  are given numbers,  $T > 0$ ,  $1 \leq p_2 < \infty$ .

**Problem C.** To pose Problem C, let us first define the following vectors:

$$h(t) = \text{col.}(h_1(t), h_2(t), \dots, h_n(t), \dots), \quad 0 \leq t \leq T \quad (4)$$

$$y = \text{col.}(y_1, y_2, \dots, y_n, \dots). \quad (5)$$

Assume that  $\{h_n(t)\}_{n=1}^\infty$  is a given set of continuous functions in  $[0, T]$ , and that

$\{h_n(t)\}_{n=1}^N$  is a set of linearly independent functions for any  $N$ . Also, assume that  $\int_0^T h(t) u(t) dt \in l_{p_2}$  for any  $u \in \Omega_{p_1}$ , where

$$l_{p_2} \triangleq \left\{ y \mid \left[ \sum_{n=1}^{\infty} |y_n|^{p_2} \right]^{1/p_2} < \infty \right\} \quad (6)$$

and  $p_2$  is a given number,  $1 \leq p_2 < \infty$ . This last assumption is valid if

$$\sum_{n=1}^{\infty} \left[ \int_0^T |h_n(t)|^{q_1} dt \right]^{p_2/q_1} < \infty,$$

where  $(1/p_1) + (1/q_1) = 1$ , a result which follows from Hölders [7] inequality.

Given the numbers  $p_2$  and  $T$ ,  $T > 0$ ,  $1 \leq p_2 < \infty$ , and the vector  $y_d$ ,  $y_d \in l_{p_2}$ , the problem is to choose a control  $u^* \in \Omega_{p_1}$  such:  $C(u) \geq C(u^*)$  for any  $u \in \Omega_{p_1}$ , where

$$C(u) \triangleq \left[ \sum_{n=1}^{\infty} \left| y_{dn} - \int_0^T h_n(t) u(t) dt \right|^{p_2} \right]^{1/p_2}. \quad (7)$$

*Problem D.* Let the following vectors be given

$$y_d = \text{col.}(y_{d1}, y_{d2}, \dots, y_{dn}, \dots) \quad (8)$$

where the sequence  $\{y_{dn}\}_{n=1}^{\infty}$  is bounded, and

$$h(t) = \text{col.}(h_1(t), h_2(t), \dots, h_n(t), \dots), \quad 0 \leq t \leq T, \quad (9)$$

where  $\{h_n(t)\}_{n=1}^N$  is a set of linearly independent and continuous functions for any  $N$ , and:  $\max_{0 \leq t \leq T} |h_n(t)| \leq M < \infty$  for any  $n$ .

Assume also that

$$\lim_{n \rightarrow \infty} y_{dn} = 0 \quad (10)$$

and

$$\lim_{n \rightarrow \infty} \left[ \int_0^T |h_n(t)|^{q_1} dt \right]^{1/q_1} = 0 \quad (11)$$

where  $(1/p_1) + (1/q_1) = 1$ . The problem is to choose a control  $u^* \in \Omega_{p_1}$  such that:  $D(u) \geq D(u^*)$  for any  $u \in \Omega_{p_1}$ , where

$$D(u) \triangleq \sup_n \left| y_{dn} - \int_0^T h_n(t) u(t) dt \right| \quad (12)$$

and  $T > 0$  is a given number.

## 3. A MAXIMUM PRINCIPLE

In this section, a maximum principle which can be applied to optimal control problems, in linear systems where the cost functions to be minimized are norms, is given.

Let  $B_1$  and  $B_2$  be normed linear spaces. Let  $\|\cdot\|_1$  denote the norm on  $B_1$  and  $\|\cdot\|_2$  denote the norm on  $B_2$ . Let  $B_2^*$  be the conjugate space of  $B_2$ , and let  $\|\cdot\|$  denote the norm on that space. Denote by  $\Omega$  the unit sphere of  $B_1$ . Assume that  $S$  is a continuous mapping from  $\Omega$  into  $B_2$  which satisfies

$$\lambda Su_1 + (1 - \lambda) Su_2 = S(\lambda u_1 + (1 - \lambda) u_2) \quad (13)$$

for any  $u_1, u_2 \in \Omega$  and  $0 \leq \lambda \leq 1$ .

From (13) it follows that the set  $K$ , where

$$K \triangleq \{x \in B_2 \mid x = Su, u \in \Omega\}, \quad (14)$$

is a convex set.

Let there be given an element  $y_d, y_d \in B_2$ . By defining the functional

$$g(u) \triangleq \|y_d - Su\|_2 \quad (15)$$

the following problem is posed:

Choose  $u^* \in \Omega$  such that  $g(u) \geq g(u^*)$  for any  $u \in \Omega$ .

**THEOREM 1.** *Assume that  $g(u) > 0$  for any  $u \in \Omega$ . Then,  $u^* \in \Omega$  will be an optimal control (i.e.,  $g(u) \geq g(u^*)$  for any  $u \in \Omega$ ), if and only if there exists a linear functional  $L \in B_2^*$  such that the following conditions are satisfied,*

$$\text{I.} \quad \|L\| = 1 \quad (16)$$

$$\text{II.} \quad L(Su^*) = \sup_{u \in \Omega} L(Su) \quad (17)$$

$$\text{III.} \quad L(y_d - Su^*) = \|y_d - Su^*\|_2. \quad (18)$$

*Proof.* Theorem 1 is an immediate result of [5] (Theorem 2.5).

Theorem 1 is considered here as a maximum principle.

**THEOREM 2.**  *$g(u^*) = 0$  for  $u^* \in \Omega$ , if and only if  $u^*$  is a solution of*

$$y_d = Su. \quad (19)$$

*Proof.* The proof follows from the definition of  $g(u)$ , (15).

In the following, the normed linear space  $B_1$  will be the Banach space  $L_{p_1}(0, T)$ , where  $p_1$  is a given number,  $1 < p_1 \leq \infty$ , and

$$L_{p_1}(0, T) \triangleq \{u \mid \|u\|_1 < \infty\} \quad (20)$$

where

$$\|u\|_1 \triangleq \left[ \int_0^T |u(t)|^{p_1} dt \right]^{1/p_1}.$$

**THEOREM 3.** For  $1 < p_1 < \infty$  there exists always a control  $u^* \in \Omega_{p_1}$  such that:  $g(u) \geq g(u^*)$  for any  $u \in \Omega_{p_1}$ .

*Proof.* The set  $\Omega_{p_1}$ ,  $1 < p_1 < \infty$ , is weakly compact and  $g(u)$  is a weakly continuous functional on  $L_{p_1}(0, T)$ . Hence, it follows from [7] that  $g(u)$  assumes its minimum value on  $\Omega_{p_1}$ .

In the next section, necessary and sufficient conditions upon the optimal controls, for the previously posed problems, will be derived by the use of Theorem 1.

#### 4. CONDITIONS UPON THE OPTIMAL CONTROLS

**THEOREM 4.** For a control  $u^* \in \Omega_{p_1}$  to satisfy:  $A(u) \geq A(u^*) > 0$  for any  $u \in \Omega_{p_1}$ , it is necessary and sufficient that there be a solution  $(\psi, u)$  to equations (21) and (22).

$$\psi(x, t) = \frac{|y_d(x, t) - y(x, t; u)|^{p_2-1}}{\left[ \int_0^T \int_0^1 |y_d(x, t) - y(x, t; u)|^{p_2} dx dt \right]^{1/q_2}} \text{sign}(y_d(x, t) - y(x, t; u))^1$$

$$0 \leq x \leq 1, 0 \leq t \leq T \quad (21)$$

$$u(t) = \frac{\left| \int_t^T \int_0^1 \psi(x, s) h(x, s, t) dx ds \right|^{q_1-1}}{\left[ \int_0^T \left| \int_t^T \int_0^1 \psi(x, s) h(x, s, t) dx ds \right|^{q_1} dt \right]^{1/p_1}} \text{sign} \int_t^T \int_0^1 \psi(x, s) h(x, s, t) dx ds$$

$$0 \leq t \leq T \quad (22)$$

where

$$\frac{1}{p_1} + \frac{1}{q_1} = 1 \quad \text{and} \quad \frac{1}{p_2} + \frac{1}{q_2} = 1.$$

<sup>1</sup> In this paper:  $\text{sign } c \triangleq \begin{cases} c/|c| & c \neq 0 \\ \text{arbitrary} & c = 0 \end{cases}$

If there exists a solution  $(\psi, u)$  to (21) and (22), then  $u^* = u$ .

*Proof.* The proof is based on Theorem 1.

Here, the normed linear space  $B_2$  will be the space  $L_{p_2}[D_2]$ , where

$$L_{p_2}[D_2] \triangleq \{y \mid \|y\|_2 < \infty\}, \quad (23)$$

and

$$\|y\|_2 \triangleq \left[ \int_0^T \int_0^1 |y(x, t)|^{p_2} dx dt \right]^{1/p_2}.$$

The mapping  $S$  from  $\Omega_{p_1}$  into  $B_2$  will be defined by

$$Su \triangleq y(\cdot, \cdot; u) \quad u \in \Omega_{p_1} \quad (24)$$

where  $y(x, t; u)$ ,  $(x, t) \in D_2$  is given by (1), and the convex set  $K$  is given by

$$K \triangleq \{y \mid y = Su, u \in \Omega_{p_1}\}. \quad (25)$$

Now, since every linear continuous functional  $L$  on  $L_{p_2}[D_2]$  can be written as

$$L(y) = \int_0^T \int_0^1 \psi(x, t) y(x, t) dx dt \quad (26)$$

where  $y \in L_{p_2}[D_2]$  and  $\psi \in L_{q_2}[D_2]$ ,  $(1/p_2) + (1/q_2) = 1$ ,  $1 \leq p_2 < \infty$ , then, for any  $u \in \Omega_{p_1}$  the operation of  $L$  on  $Su$  yields:

$$L(Su) = \int_0^T \int_0^1 \psi(x, s) y_0(x, s) dx ds + \int_0^T u(t) \int_t^T \int_0^1 \psi(x, s) h(x, s, t) ds dx dt. \quad (27)$$

From (27) and Hölder's [7] inequality it follows that condition II of Theorem 1 is satisfied if and only if  $u^* = u$  where  $u$  is given by (22).

Condition III of Theorem 1 can be written here as

$$\int_0^T \int_0^1 \psi(x, t) [y_a(x, t) - y(x, t; u^*)] dx dt = \left[ \int_0^T \int_0^1 |y_a(x, t) - y(x, t; u^*)|^{p_2} dx dt \right]^{1/p_2} \quad (28)$$

and this equation is satisfied if and only if  $\psi$  is given by (21), in which  $u = u^*$  is inserted.

Also, if  $\psi$  is given by (21) then condition I of Theorem 1 is satisfied.

THEOREM 5. For a control  $u^* \in \Omega_{p_1}$  to satisfy:  $B(u) \geq B(u^*) > 0$  for any  $u \in \Omega_{p_1}$ , it is necessary and sufficient that there be a solution  $(\psi, u)$  to equations (29) and (30).

$$\psi(x) = \frac{|y_d(x) - y(x, T; u)|^{p_2-1} \operatorname{sign}(y_d(x) - y(x, T; u))}{\left[ \int_0^1 |y_d(x) - y(x, T; u)|^{p_2} dx \right]^{1/q_2}} \quad 0 \leq x \leq 1 \quad (29)$$

$$u(t) = \frac{\left| \int_0^1 \psi(x) h(x, T, t) dx \right|^{q_1-1} \operatorname{sign} \int_0^1 \psi(x) h(x, T, t) dx}{\left[ \int_0^T \left| \int_0^1 \psi(x) h(x, T, t) dx \right|^{q_1} dt \right]^{1/p_1}} \quad 0 \leq t \leq T \quad (30)$$

where

$$\frac{1}{p_1} + \frac{1}{q_1} = 1 \quad \text{and} \quad \frac{1}{p_2} + \frac{1}{q_2} = 1.$$

If there exists a solution  $(\psi, u)$  to (29) and (30), then  $u^* = u$ .

*Proof.* The proof is similar to the proof of Theorem 4.

THEOREM 6. Assume that  $[\sum_{n=1}^{\infty} |h_n(t)|^{p_2}]^{q_1/p_2}$  is integrable.

Then, for a control  $u^* \in \Omega_{p_1}$  to satisfy:  $C(u) \geq C(u^*) > 0$  for any  $u \in \Omega_{p_1}$ , it is necessary and sufficient that there be a solution  $(c, u)$  to equations (31) and (32).

$$c_i = \frac{\left| y_{di} - \int_0^T h_i(t) u(t) dt \right|^{p_2-1} \operatorname{sign} \left( y_{di} - \int_0^T h_i(t) u(t) dt \right)}{\left[ \sum_{n=1}^{\infty} \left| y_{dn} - \int_0^T h_n(t) u(t) dt \right|^{p_2} \right]^{1/q_2}} \quad i = 1, 2, \dots \quad (31)$$

$$u(t) = \frac{\left| \sum_{n=1}^{\infty} c_n h_n(t) \right|^{q_1-1} \operatorname{sign} \sum_{n=1}^{\infty} c_n h_n(t)}{\left[ \int_0^T \left| \sum_{n=1}^{\infty} c_n h_n(t) \right|^{q_1} dt \right]^{1/p_1}} \quad 0 \leq t \leq T \quad (32)$$

where

$$\frac{1}{p_1} + \frac{1}{q_1} = 1 \quad \text{and} \quad \frac{1}{p_2} + \frac{1}{q_2} = 1.$$

If there exists a solution  $(c, u)$  to (31) and (32), then  $u^* = u$ .

*Proof.* The proof is based on Theorem 1.

Here, the normed linear space  $B_2$  will be the space  $l_{p_2}$ , where

$$l_{p_2} \triangleq \{y \mid \|y\|_2 < \infty\} \quad (33)$$

and

$$\|y\|_2 \triangleq \left[ \sum_{n=1}^{\infty} |y_n|^{p_2} \right]^{1/p_2}.$$

The mapping  $S$  from  $\Omega_{p_1}$  into  $B_2$  will be defined by

$$Su \triangleq \int_0^T h(t) u(t) dt \quad u \in \Omega_{p_1} \quad (34)$$

where  $h(t)$  is given by (5), and the convex set  $K$  is given by

$$K \triangleq \{y \mid y = Su, u \in \Omega_{p_1}\}, \quad (35)$$

Now, since every linear continuous functional  $L$  on  $l_{p_2}$  can be written as

$$L(y) = \sum_{n=1}^{\infty} c_n y_n \quad (36)$$

where  $y \in l_{p_2}$  and  $c \in l_{q_2}$ ,  $(1/p_2) + (1/q_2) = 1$ ,  $1 \leq p_2 < \infty$ , then, for any  $u \in \Omega_{p_1}$  the operation of  $L$  on  $Su$  yields:

$$L(Su) = \int_0^T \sum_{n=1}^{\infty} c_n h_n(t) u(t) dt. \quad (37)$$

From (37) and Hölder's inequality<sup>3</sup> it follows that condition II of Theorem 1 is satisfied if and only if  $u^* = u$  where  $u$  is given by (32).

Condition III of Theorem 1 can be written here as

$$\sum_{n=1}^{\infty} c_n \left( y_{dn} - \int_0^T h_n(t) u^*(t) dt \right) = \left[ \sum_{n=1}^{\infty} \left| y_{dn} - \int_0^T h_n(t) u^*(t) dt \right|^{p_2} \right]^{1/p_2} \quad (38)$$

and this equation is satisfied if and only if  $c$  is given by (31), in which  $u = u^*$  is inserted. Also, if  $c$  is given by (31) then condition I of Theorem 1 is satisfied.

<sup>3</sup> Here, Bepo-Levi theorem [8] is used.

<sup>4</sup> Using the fact that  $\left[ \sum_{n=1}^{\infty} |h_n(t)|^{p_2} \right]^{q_1/p_2}$  is integrable.



*Remark.* The case where  $C(u^*) = 0$  for  $u^* \in \Omega_{p_1}$ , is treated by A. G. Butkovskii and L. N. Poltavskii [9], by considering it as a problem in the theory of moments.

**THEOREM 7.** For a control  $u^* \in \Omega_{p_1}$  to satisfy:  $D(u) \geq D(u^*) > 0$  for any  $u \in \Omega_{p_1}$ , it is necessary and sufficient that there be a solution  $(d, u)$  to equations (39) and (40).

$$d_i = \begin{cases} 0 & i \notin J \\ k_i \operatorname{sign} \left( y_{di} - \int_0^T h_i(t) u(t) dt \right) & i \in J \end{cases} \quad (39)$$

where  $k_i \geq 0$  for any  $i$ , and  $\sum_{i \in J} k_i = 1$ , and  $i \in J$  if  $\sup_n |y_{an} - \int_0^T h_n(t) u(t) dt|$  is assumed for  $n = i$ .

$$u(t) = \frac{\left| \sum_{n=1}^{\infty} d_n h_n(t) \right|^{q_1-1}}{\left[ \int_0^T \left| \sum_{n=1}^{\infty} d_n h_n(t) \right|^{q_1} dt \right]^{1/p_1}} \operatorname{sign} \sum_{n=1}^{\infty} d_n h_n(t) \quad 0 \leq t \leq T. \quad (40)$$

If there exists a solution  $(d, u)$  to (39) and (40) then  $u^* = u$ .

*Proof.* The proof is based on Theorem 1.

Here, the normed linear space  $B_2$  will be the space  $c_0$ , where

$$c_0 \triangleq \{y = (y_1, y_2, \dots, y_n, \dots) \mid \{y_n\} \text{ are bounded and } \lim_{n \rightarrow \infty} y_n = 0\}, \quad (41)$$

and

$$\|y\|_2 \triangleq \sup_n |y_n|.$$

The mapping  $S$  from  $\Omega_{p_1}$  into  $B_2$  will be defined by

$$Su \triangleq \int_0^T h(t) u(t) dt \quad u \in \Omega_{p_1} \quad (42)$$

where  $h(t)$  is given by (9), and the convex set  $K$  is given by

$$K \triangleq \{y \mid y = Su, u \in \Omega_{p_1}\}. \quad (43)$$

Now, since every linear continuous functional  $L$  on  $c_0$  can be written as

$$L(y) = \sum_{n=1}^{\infty} d_n y_n \quad (44)$$

where  $y \in c_0$  and  $d \in l_1$ , then, for any  $u \in \Omega_{p_1}$  the operation of  $L$  on  $Su$  yields:

$$L(Su) = \int_0^T \sum_{n=1}^{\infty} d_n h_n(t) u(t) dt. \quad (45)$$

From (45) and Hölder's inequality it follows that condition II of Theorem 1 is satisfied if and only if  $u^* = u$  where  $u$  is given by (40).

Condition III of Theorem 1 can be written here as

$$\sum_{n=1}^{\infty} d_n \left( y_{dn} - \int_0^T h_n(t) u^*(t) dt \right) = \sup_n \left| y_{dn} - \int_0^T h_n(t) u^*(t) dt \right|, \quad (46)$$

and this equation is satisfied if and only if  $d$  is given by (39), in which  $u = u^*$  is inserted.

Also, if  $d$  is given by (39) then condition I of Theorem 1 is satisfied.

*Remark.* Since  $D(u)$  is not a differentiable functional, it follows that Butkovskii's maximum principle ([1], [2]), and Demyabov and Rubinov's theorem [3], are not applicable to Problem D.

## 5. PONTRYAGIN'S MAXIMUM PRINCIPLE

In this section it is shown, that for optimal control problems in linear lumped parameter systems, where the cost functions are norms, Theorem 1 is an extension of Pontryagin's maximum principle [6].

Let there be given the linear system:

$$\dot{x}(t) = A(t)x(t) + G(t)u(t) \quad x(0) = x_0 \quad (47)$$

where  $A(t)$  is a  $n \times n$  matrix,  $G(t)$  a  $n$ -dimensional vector and  $u$  is a scalar function. All the elements of  $A(t)$  and  $G(t)$  are real and continuous in  $[0, T]$ , and  $u \in \Omega_{p_1}$ ,  $1 < p_1 \leq \infty$ .

For the system (47) two optimal control problems will be posed. One, (Problem E), for which Pontryagin's maximum principle is applicable, and the other, (Problem F), for which Pontryagin's maximum principle is not applicable. To these two problems, Theorem 1 will be applied.

*Problem E.* Choose  $u^* \in \Omega_{p_1}$  such that  $E(u) \geq E(u^*)$  for any  $u \in \Omega_{p_1}$ , where

$$E(u) \triangleq \left[ \int_0^T \sum_{i=1}^n |x_i(t; u)|^{p_2} dt \right]^{1/p_2} \quad 1 \leq p_2 < \infty. \quad (48)$$

For any  $u \in \Omega_{p_1}$  the solution to (47) is given by

$$x(t; u) = \phi(t, 0) x_0 + \int_0^t \phi(t, s) G(s) u(s) ds \quad (49)$$

where  $\phi(t, s)$  is the transition matrix of the system (47).

Similarly to the proof of Theorem 4, the following theorem can be proved.

**THEOREM 8.** *For a control  $u^* \in \Omega_{p_1}$  to satisfy:  $E(u) \geq E(u^*) > 0$  for any  $u \in \Omega_{p_1}$ , it is necessary and sufficient that there be a solution  $(\lambda, u)$  to equations (50) and (51).*

$$\lambda_i(t) = \frac{- |x_i(t; u)|^{p_2-1} \text{sign } x_i(t; u)}{\left[ \int_0^T \sum_{i=1}^n |x_i(t; u)|^{p_2} dt \right]^{1/q_2}} \quad 0 \leq t \leq T, i = 1, 2, \dots, n \quad (50)$$

$$u(t) = \frac{\left| \int_t^T \lambda'(s) \phi(s, t) G(t) ds \right|^{q_1-1}}{\left[ \int_0^T \left| \int_t^T \lambda'(s) \phi(s, t) G(t) ds \right|^{q_1} dt \right]^{1/p_1}} \text{sign } \int_t^T \lambda'(s) \phi(s, t) G(t) ds \quad 0 \leq t \leq T \quad (51)$$

where

$$\frac{1}{p_1} + \frac{1}{q_1} = 1 \quad \text{and} \quad \frac{1}{p_2} + \frac{1}{q_2} = 1.$$

If there exists a solution  $(\lambda, u)$  to (50) and (51) then  $u^* = u$ .

Now, two new vectors,  $\tilde{\lambda}$  and  $p$  will be defined for the purpose of showing that Theorem 8 is identical to Pontryagin's results:

$$\tilde{\lambda}_i(t) \triangleq - p_2 |x_i(t; u)|^{p_2-1} \text{sign } x_i(t; u) \quad i = 1, 2, \dots, n, 0 \leq t \leq T \quad (52)$$

$$p(t) \triangleq \int_t^T \phi'(s, t) \tilde{\lambda}(s) ds \quad 0 \leq t \leq T. \quad (53)$$

From (53) it follows that  $p$  is the solution of the system.

$$\dot{p}(t) = -A'(t)p(t) - \tilde{\lambda}(t) \quad p(T) = 0. \quad (54)$$

By the insertion of (50) into (51), Theorem 8 can be rewritten in the following equivalent form:

THEOREM 9. For a control  $u^* \in \Omega_{p_1}$  to satisfy:  $E(u) \geq E(u^*) > 0$  for any  $u \in \Omega_{p_1}$ , it is necessary and sufficient that  $u^*$  be a solution of the equations

$$u(t) = \frac{|G'(t) p(t)|^{q_1-1}}{\left[\int_0^T |G'(t) p(t)|^{q_1} dt\right]^{1/p_1}} \text{sign}(G'(t) p(t)) \quad 0 \leq t \leq T \quad (55)$$

$$\dot{x}(t) = A(t)x(t) + G(t)u(t) \quad x(0) = x_0 \quad (56)$$

$$p(t) = A'(t)p(t) - \tilde{\lambda}(t) \quad p(T) = 0. \quad (57)$$

Theorem 9 is identical to the necessary and sufficient condition obtained from Pontryagin's maximum principle [6], for Problem E.

Problem F. Choose  $u^* \in \Omega_{p_1}$  such that  $F(u) \geq F(u^*)$  for any  $u \in \Omega_{p_1}$ , where

$$F(u) \triangleq \max_{1 \leq i \leq n} |x_i(T; u)|, \quad (58)$$

and  $T$  is a given positive number.

Since  $F(u)$  is not a differentiable functional, it follows that Pontryagin's maximum principle [6] can not be applied to Problem F. But, using Theorem 1, the following theorem can be proved.

THEOREM 10. For a control  $u^* \in \Omega_{p_1}$  to satisfy:  $F(u) \geq F(u^*) > 0$  for any  $u \in \Omega_{p_1}$ , it is necessary and sufficient that there be a solution  $(f, u)$  to equations (59) and (60).

$$f_i = \begin{cases} 0 & i \notin J \\ k_i \text{sign } x_i(T; u) & i \in J \end{cases} \quad (59)$$

where  $k_i \geq 0$ ,  $1 \leq i \leq n$ , and  $\sum_{i \in J} k_i = 1$ , and  $i \in J$  if  $\max_{1 \leq j \leq n} |x_j(T; u)|$  is assumed for  $j = i$ .

$$u(t) = \frac{|f' \phi(T, t) G(t)|^{q_1-1}}{\left[\int_0^T |f' \phi(T, t) G(t)|^{q_1} dt\right]^{1/p_1}} \text{sign}(f' \phi(T, t) G(t)). \quad (60)$$

If there exists a solution  $(f, u)$  to (59) and (60) then  $u^* = u$ .

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